(cosθ, sinθ)

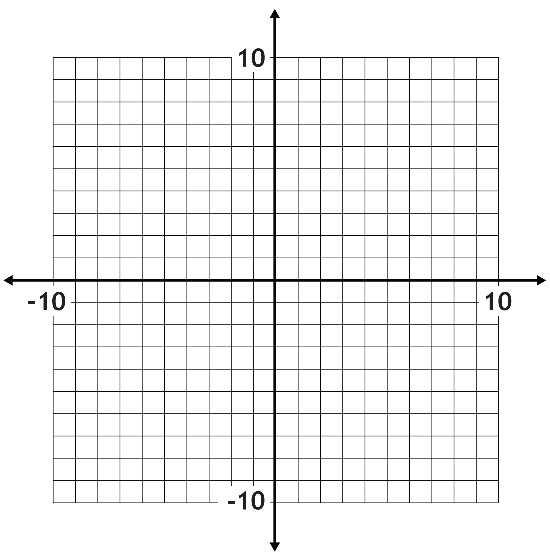
Trigonometry
LESSON TWO - *The Unit Circle*
Lesson Notes

Example 1 Introduction to Circle Equations.

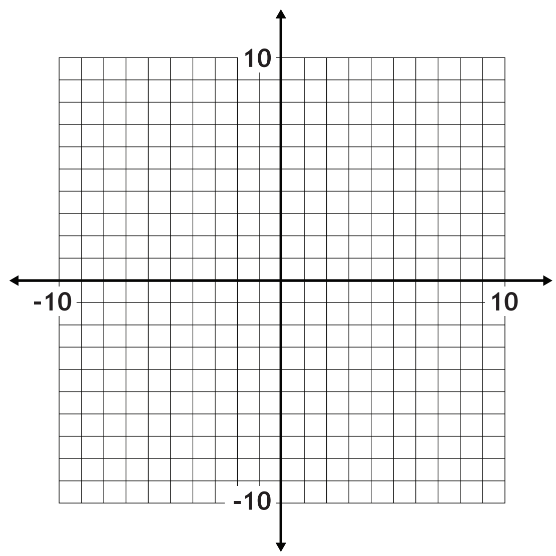
Equation of a Circle

a) A circle centered at the origin can be represented by the relation $x^2 + y^2 = r^2$, where r is the radius of the circle. Draw each circle:

i) $x^2 + y^2 = 4$



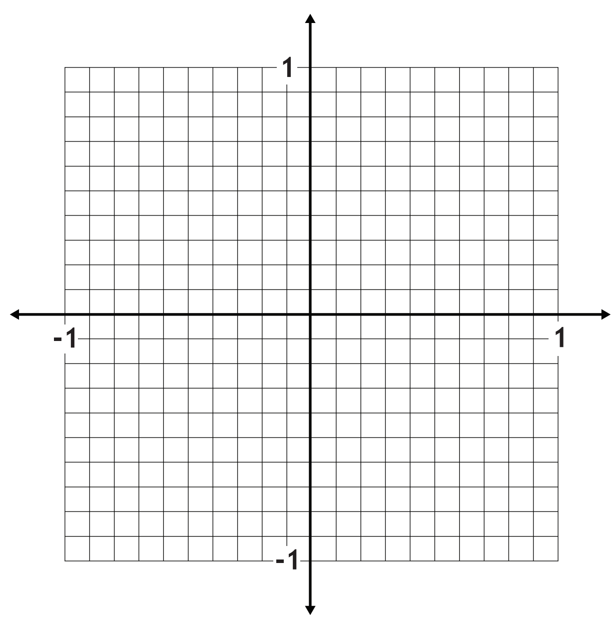
ii) $x^2 + y^2 = 49$



b) A circle centered at the origin with a radius of 1 has the equation $x^2 + y^2 = 1$. This special circle is called the *unit circle*. Draw the unit circle and determine if each point exists on the circumference of the unit circle.

i) (0.6, 0.8)

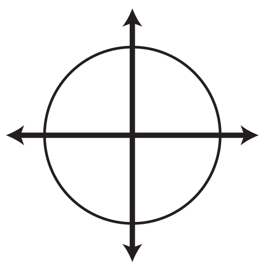
ii) (0.5, 0.5)

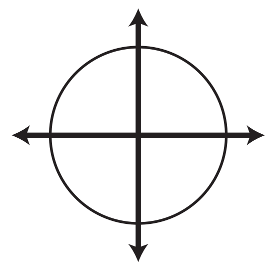


c) Using the equation of the unit circle, $x^2 + y^2 = 1$, find the unknown coordinate of each point. Is there more than one unique answer?

i) $\left(\frac{1}{2}, y\right)$

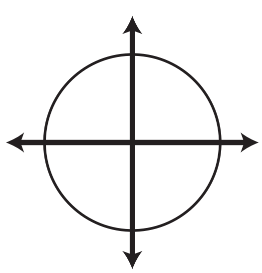
ii) $\left(x, \frac{\sqrt{3}}{2}\right)$, quadrant II.

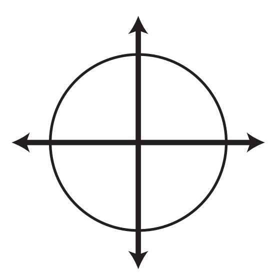




iii) $(-1, y)$

iv) $\left(x, -\frac{\sqrt{2}}{2}\right)$, $\cos\theta > 0$.





$(\cos\theta, \sin\theta)$

Trigonometry
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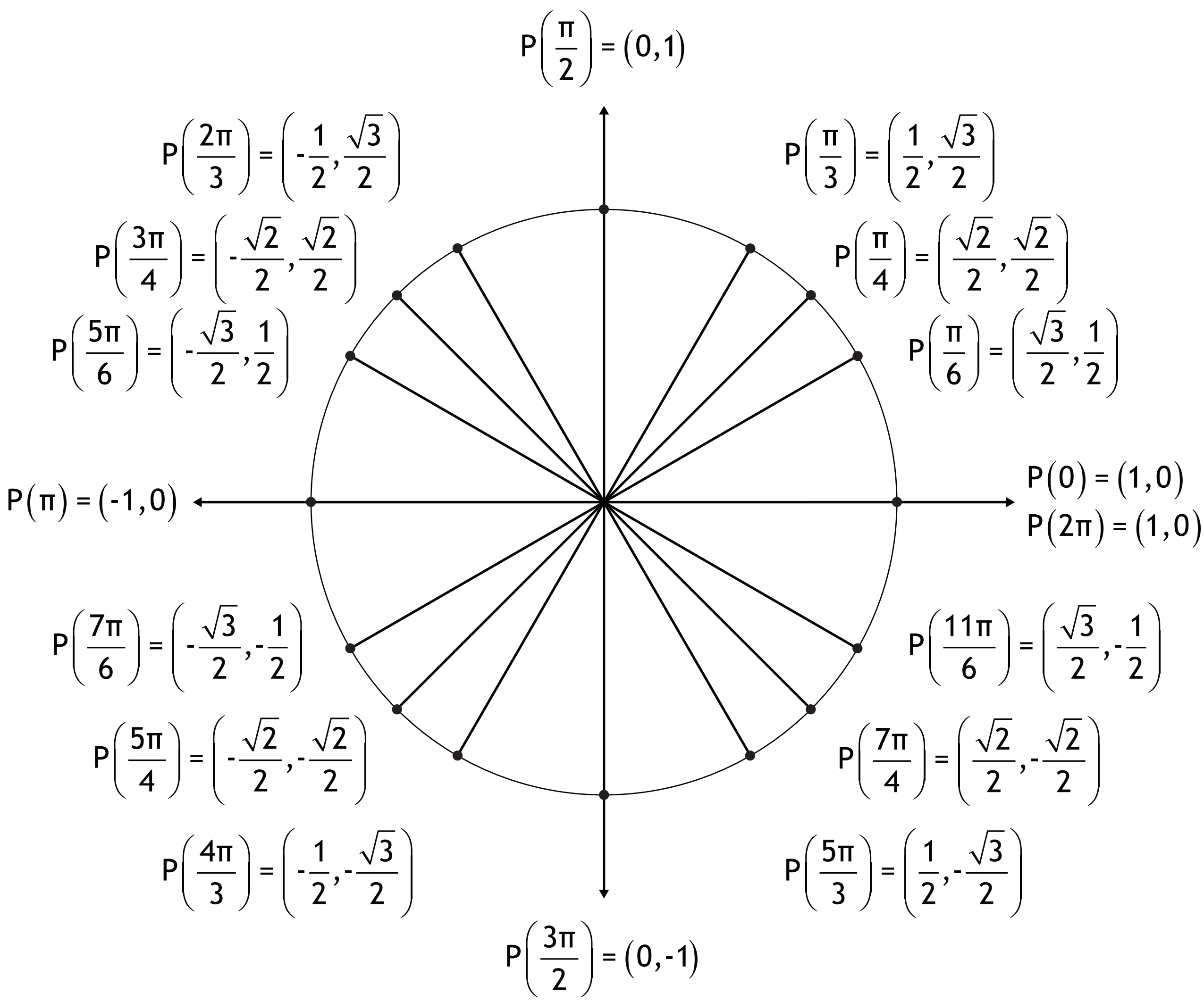
Example 2 The Unit Circle.

The Unit Circle

The following diagram is called *the unit circle*. Commonly used angles are shown as radians, and their exact-value coordinates are in brackets. Take a few moments to memorize this diagram. When you are done, use the blank unit circle on the next page to practice drawing the unit circle from memory.



questions on next page.

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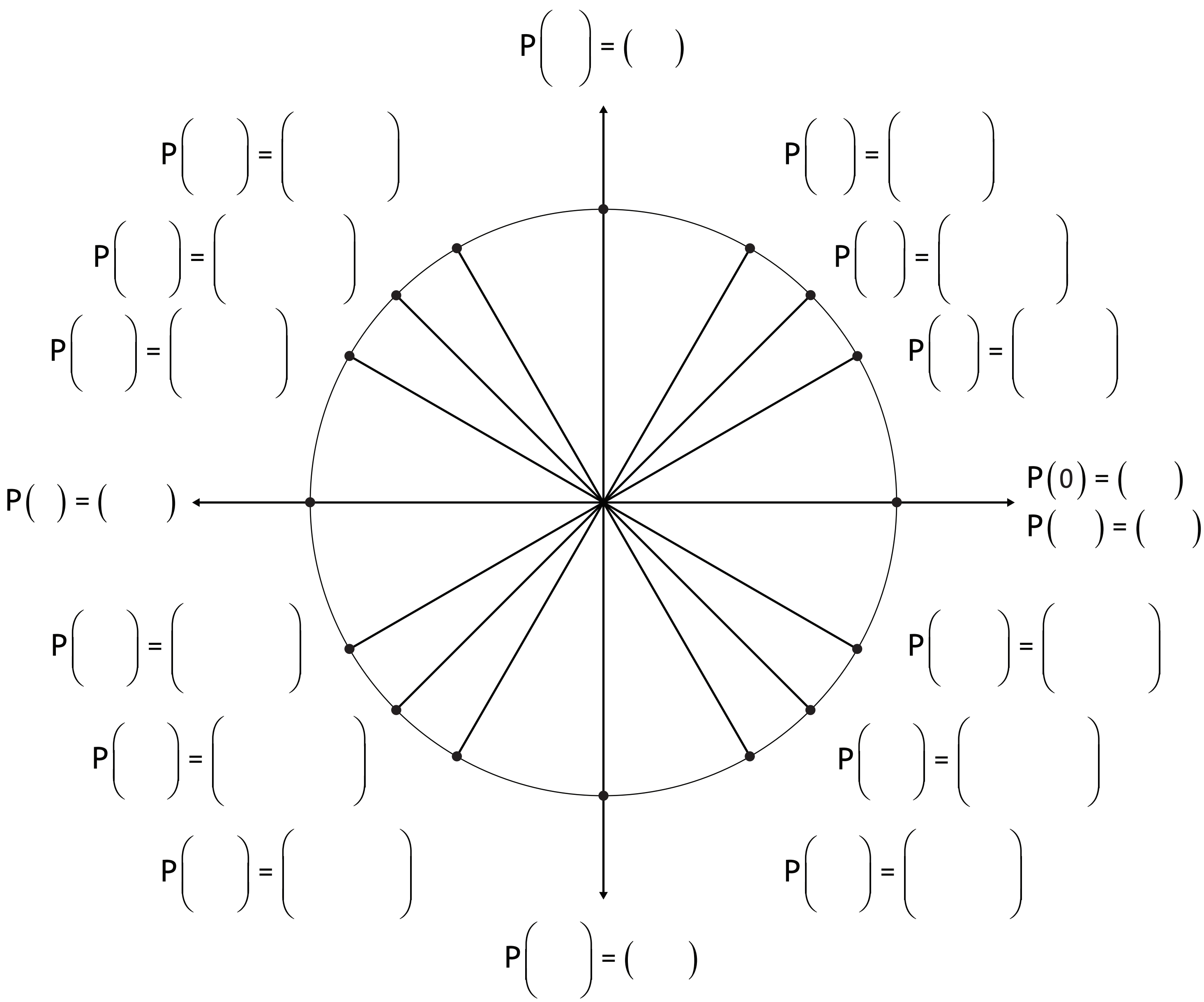
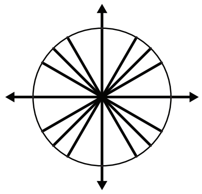
$(\cos\theta, \sin\theta)$

a) What are some useful tips to memorize the unit circle?

b) Draw the unit circle from memory using a partially completed template.





$(\cos\theta, \sin\theta)$

Example 3 Use the unit circle to find the exact value of each trigonometric ratio.

Finding Primary Trigonometric Ratios with the Unit Circle

a) $\sin \frac{\pi}{6}$

b) $\cos 180^\circ$

c) $\cos \frac{3\pi}{4}$

d) $\sin \frac{11\pi}{6}$

e) $\sin 0$

f) $\cos -\frac{\pi}{2}$

g) $\sin \frac{4\pi}{3}$

h) $\cos -120^\circ$

Example 4 Use the unit circle to find the exact value of each trigonometric ratio.

a) $\cos 420^\circ$

b) $-\cos 3\pi$

c) $\sin \frac{13\pi}{6}$

d) $\cos -\frac{2\pi}{3}$

e) $\sin -\frac{5\pi}{2}$

f) $-\sin \frac{9\pi}{4}$

g) $\cos^2 (-840^\circ)$

h) $\cos -\frac{7\pi}{3}$

Trigonometry
LESSON TWO - *The Unit Circle*
Lesson Notes

$(\cos\theta, \sin\theta)$

Example 5 Other Trigonometric Ratios.

The unit circle contains values for $\cos\theta$ and $\sin\theta$ only. The other four trigonometric ratios can be obtained using the identities on the right.

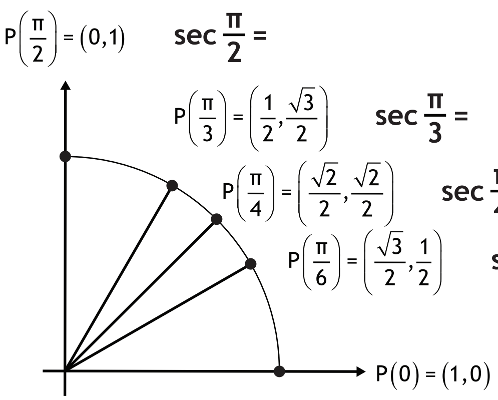
Given angles from the first quadrant of the unit circle, find the exact values of $\sec\theta$ and $\csc\theta$.

Other Trigonometric Ratios

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

a) $\sec\theta$



$\sec\frac{\pi}{2} =$

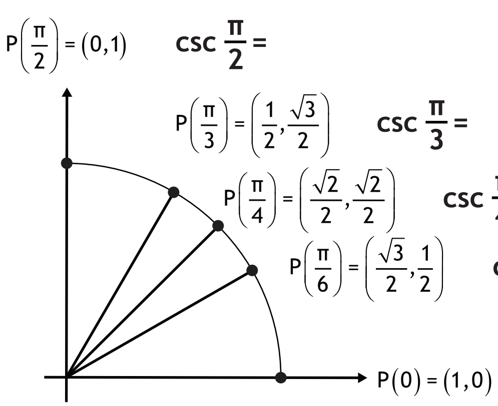
$\sec\frac{\pi}{3} =$

$\sec\frac{\pi}{4} =$

$\sec\frac{\pi}{6} =$

$\sec 0 =$

b) $\csc\theta$



$\csc\frac{\pi}{2} =$

$\csc\frac{\pi}{3} =$

$\csc\frac{\pi}{4} =$

$\csc\frac{\pi}{6} =$

$\csc 0 =$

Example 6

Other Trigonometric Ratios.

The unit circle contains values for $\cos\theta$ and $\sin\theta$ only. The other four trigonometric ratios can be obtained using the identities on the right.

Given angles from the first quadrant of the unit circle, find the exact values of $\tan\theta$ and $\cot\theta$.

Other Trigonometric Ratios

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

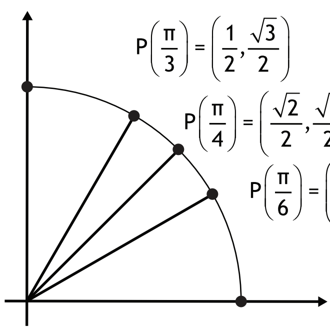
a) $\tan\theta$

$P\left(\frac{\pi}{2}\right) = (0,1)$ **$\tan\frac{\pi}{2}$ =**

$P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **$\tan\frac{\pi}{3}$ =**

$P\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **$\tan\frac{\pi}{4}$ =**

$P\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **$\tan\frac{\pi}{6}$ =**



$P(0) = (1,0)$ **$\tan 0$ =**

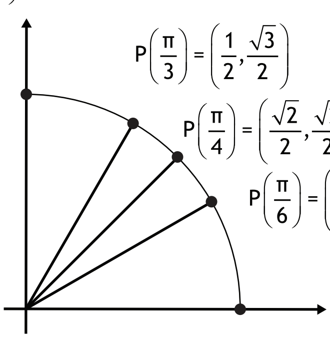
b) $\cot\theta$

$P\left(\frac{\pi}{2}\right) = (0,1)$ **$\cot\frac{\pi}{2}$ =**

$P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **$\cot\frac{\pi}{3}$ =**

$P\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **$\cot\frac{\pi}{4}$ =**

$P\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **$\cot\frac{\pi}{6}$ =**



$P(0) = (1,0)$ **$\cot 0$ =**

Trigonometry

LESSON TWO - *The Unit Circle*

Lesson Notes

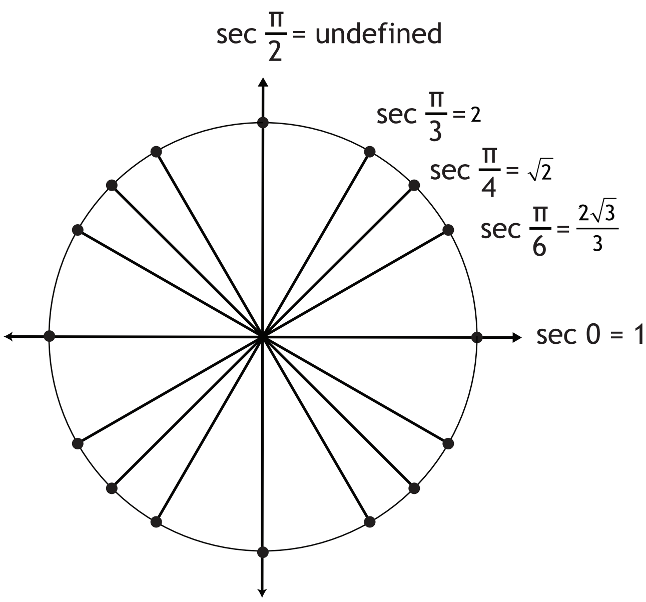
$(\cos\theta, \sin\theta)$

Example 7

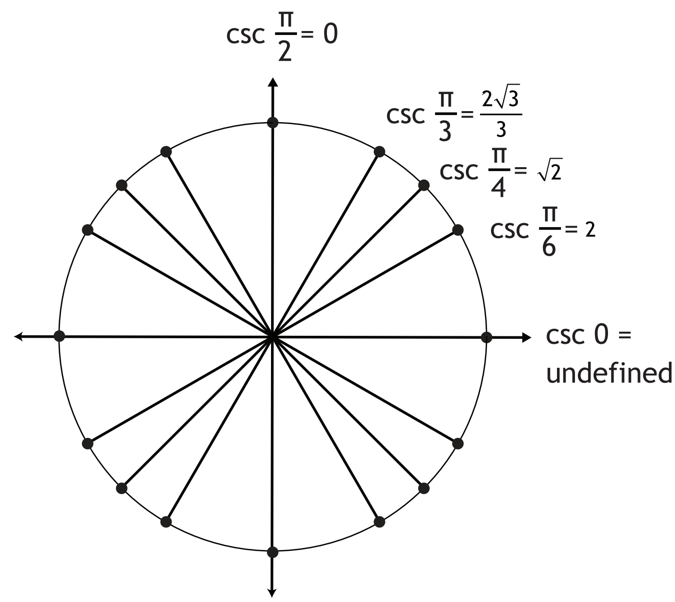
Use symmetry to fill in quadrants II, III, and IV for each unit circle.

Symmetry of the Unit Circle

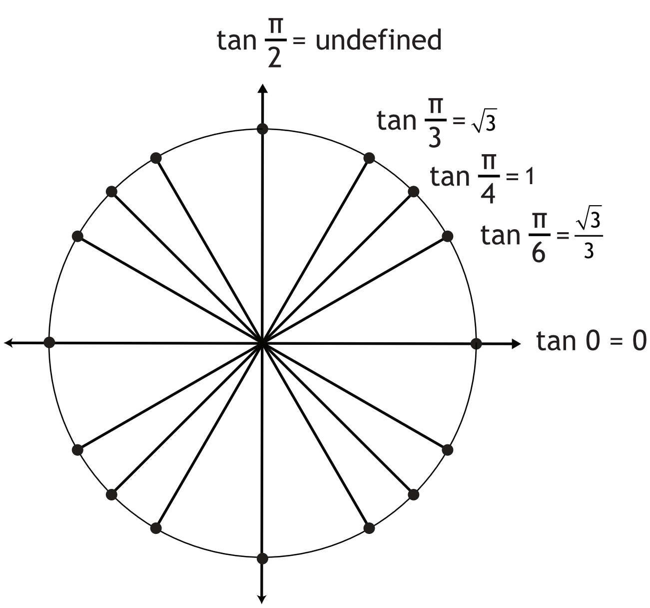
a) $\sec\theta$



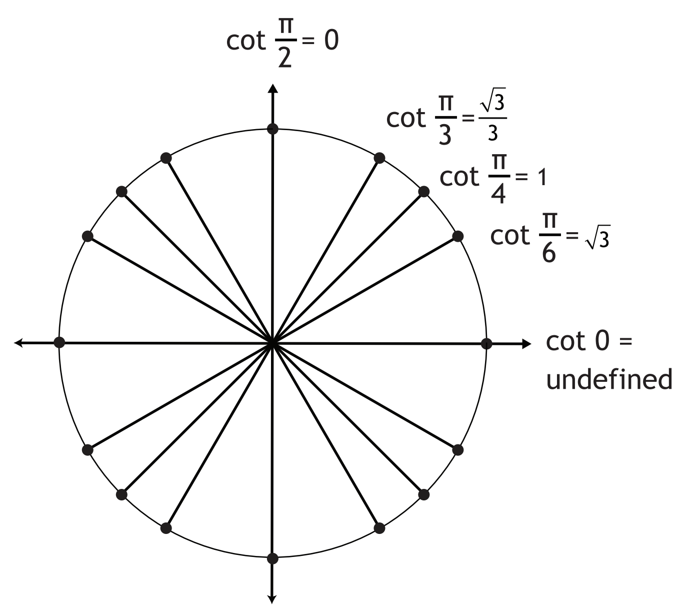
b) $\csc\theta$



c) $\tan\theta$



d) $\cot\theta$



Example 8

Find the exact value of each trigonometric ratio.

Finding Reciprocal Trigonometric Ratios with the Unit Circle

a) $\sec 120^\circ$

b) $\sec \frac{3\pi}{2}$

c) $\csc \frac{\pi}{3}$

d) $\csc -\frac{3\pi}{4}$

e) $\tan \frac{\pi}{6}$

f) $-\tan \frac{5\pi}{4}$

g) $\cot^2(270^\circ)$

h) $\cot -\frac{5\pi}{6}$

Example 9 Find the exact value of each trigonometric expression.

Evaluating Complex Expressions with the Unit Circle

a) $\sin\left(-\frac{\pi}{3}\right) + \cos\left(\frac{5\pi}{4}\right)$

b) $\cos^2\frac{\pi}{4} + \sin^2\frac{\pi}{4}$

c) $\cot^2\frac{\pi}{3} + 1$

d) $\dfrac{\sec\frac{\pi}{6}}{\tan\frac{\pi}{6} + \cot\frac{\pi}{6}}$

Example 10 Find the exact value of each trigonometric expression.

Evaluating Complex Expressions with the Unit Circle

a) $\sin\frac{\pi}{3}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6}$

b) $\cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

c) $\dfrac{2\tan\frac{\pi}{6}}{1 - \tan^2\frac{\pi}{6}}$

d) $\dfrac{\tan\frac{3\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{3\pi}{4}\tan\frac{\pi}{6}}$

Example 11

Find the exact value of each trigonometric ratio.

Finding the Trigonometric Ratios of Large Angles with the Unit Circle

a) $\csc\left(-\frac{9\pi}{2}\right)$

b) $-\tan^2\left(\frac{617\pi}{6}\right)$

c) $\sec\left(\frac{61\pi}{2}\right)$

d) $\cot\left(-1980^\circ\right)$

Example 12

Verify each trigonometric statement with a calculator. *Note: Every question in this example has already been seen earlier in the lesson.*

Evaluating Trigonometric Ratios with a Calculator

a) $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

b) $\cos^2\left(-840^\circ\right) = \frac{1}{4}$

c) $\sec\frac{3\pi}{2} = \textit{undefined}$

d) $\csc\left(-\frac{3\pi}{4}\right) = -\sqrt{2}$

e) $-\tan^2\left(\frac{617\pi}{6}\right) = -\frac{1}{3}$

f) $\cot^2\frac{\pi}{3} + 1 = \frac{4}{3}$

g) $\dfrac{\sec\frac{\pi}{6}}{\tan\frac{\pi}{6} + \cot\frac{\pi}{6}} = \frac{1}{2}$

h) $\dfrac{\tan\frac{3\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{3\pi}{4}\tan\frac{\pi}{6}} = -2 - \sqrt{3}$

Example 13 Answer each of the following questions related to the unit circle.

Coordinate Relationships on the Unit Circle

a) What is meant when you are asked to find $P\left(\frac{\pi}{3}\right)$ on the unit circle?

b) Find one positive and one negative angle such that $P(\theta) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

c) How does a half-rotation around the unit circle change the coordinates?
If $\theta = \frac{\pi}{6}$, find the coordinates of the point halfway around the unit circle.

d) How does a quarter-rotation around the unit circle change the coordinates?
If $\theta = \frac{2\pi}{3}$, find the coordinates of the point a quarter-revolution (clockwise) around the unit circle.

e) What are the coordinates of P(3)? Express coordinates to four decimal places.

Example 14 Answer each of the following questions related to the unit circle.

Circumference and Arc Length of the Unit Circle

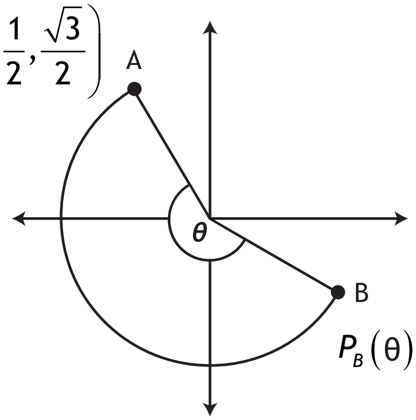
a) What is the circumference of the unit circle?

b) How is the central angle of the unit circle related to its corresponding arc length?

c) If a point on the terminal arm rotates from P(θ) = (1, 0) to P(θ) = $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, what is the arc length?

d) What is the arc length from point A to point B on the unit circle?

$P_A(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

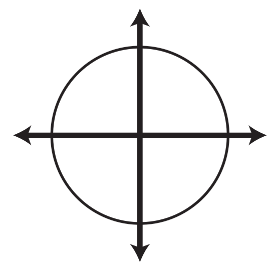


$P_B(\theta) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Example 15 Answer each of the following questions related to the unit circle.

Domain and Range of the Unit Circle

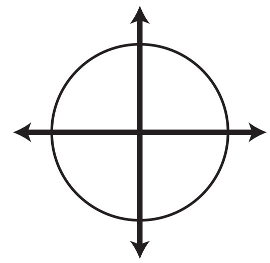
a) Is sinθ = 2 possible? Explain, using the unit circle as a reference.



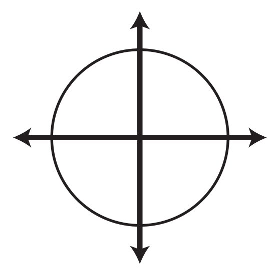
b) Which trigonometric ratios are restricted to a range of $-1 \leq y \leq 1$? Which trigonometric ratios exist outside that range?

	Range	Number Line
cosθ & sinθ		
cscθ & secθ		
tanθ & cotθ		

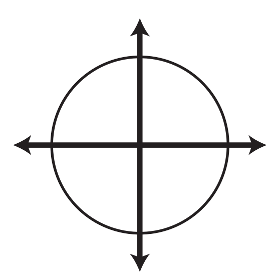
c) If $\sin\theta = -\frac{\sqrt{2}}{2}$ exists on the unit circle, how can the unit circle be used to find $\cos\theta$?
How many values for $\cos\theta$ are possible?

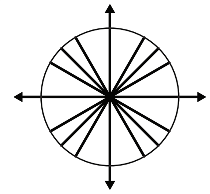


d) If $\cos\theta = \frac{3}{5}$ exists on the unit circle, how can the equation of the unit circle be used to find $\sin\theta$?
How many values for $\sin\theta$ are possible?



e) If $\cos\theta = 0$, and $0 \leq \theta < \pi$, how many values for $\sin\theta$ are possible?

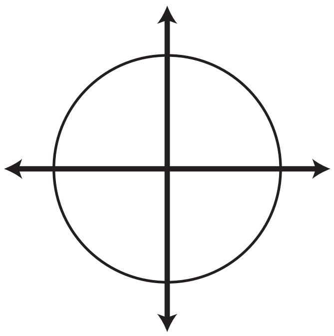




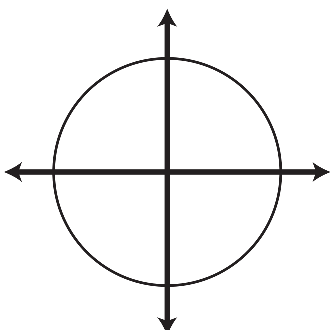
Example 16 Complete the following questions related to the unit circle.

Unit Circle Proofs

a) Use the Pythagorean Theorem to prove that the equation of the unit circle is $x^2 + y^2 = 1$.

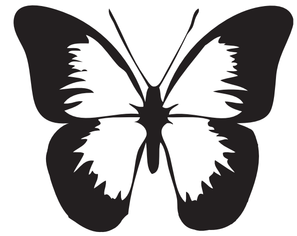


b) Prove that the point where the terminal arm intersects the unit circle, $P(\theta)$, has coordinates of $(\cos\theta, \sin\theta)$.



c) If the point $P(\theta) = \left(-\frac{40}{41}, \frac{9}{41}\right)$ exists on the terminal arm of a unit circle, find the exact values of the six trigonometric ratios. State the reference angle and standard position angle to the nearest hundredth of a degree.

Example 17



In a video game, the graphic of a butterfly needs to be rotated. To make the butterfly graphic rotate, the programmer uses the equations:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

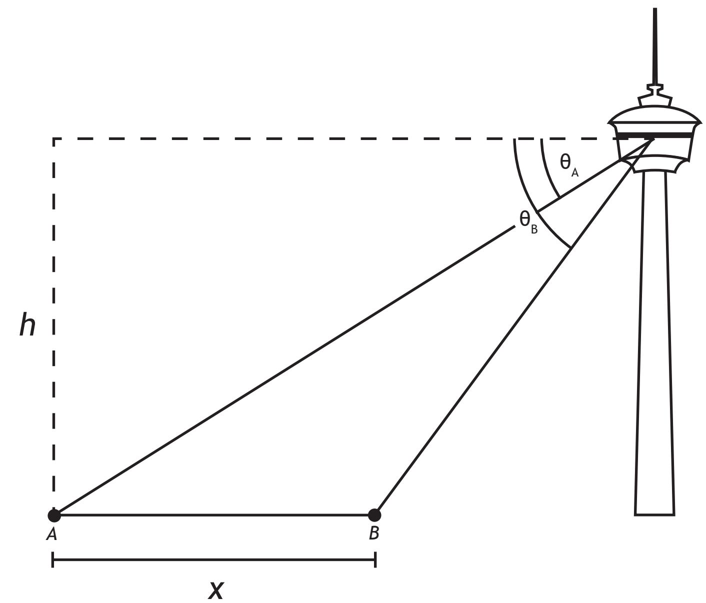
to transform each pixel of the graphic from its original coordinates, (x, y), to its new coordinates, (x', y'). Pixels may have positive or negative coordinates.

a) If a particular pixel with coordinates of (250, 100) is rotated by $\frac{\pi}{6}$, what are the new coordinates? Round coordinates to the nearest whole pixel.

b) If a particular pixel has the coordinates (640, 480) after a rotation of $\frac{5\pi}{4}$, what were the original coordinates? Round coordinates to the nearest whole pixel.

Example 18

From the observation deck of the Calgary Tower, an observer has to tilt their head θ_A down to see point A, and θ_B down to see point B.



a) Show that the height of the observation deck is $h = \dfrac{x}{\cot\theta_A - \cot\theta_B}$.

b) If $\theta_A = \dfrac{131}{900}\pi$, $\theta_B = \dfrac{61}{200}\pi$, and x = 212.92 m, how high is the observation deck above the ground, to the nearest metre?